CS 430/536
Computer Graphics I

B-Splines and NURBS

Week 5, Lecture 9

David Breen, William Regli and Maxim Peysakhov

Geometric and Intelligent Computing Laboratory

Department of Computer Science

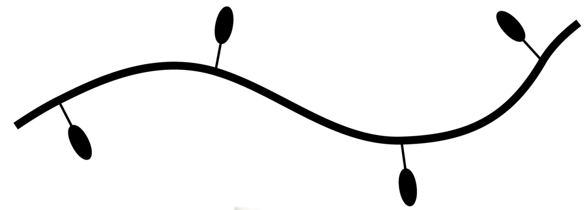
Drexel University

`http://gicl.cs.drexel.edu`

Outline

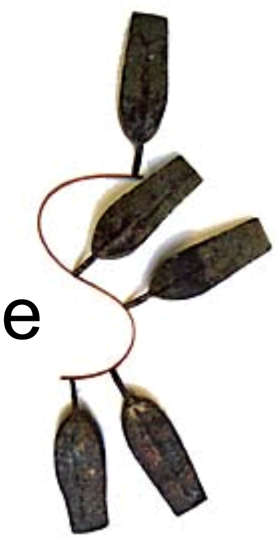
- Types of Curves
  - Splines
  - B-splines
  - NURBS
- Knot sequences
- Effects of the weights

Splines



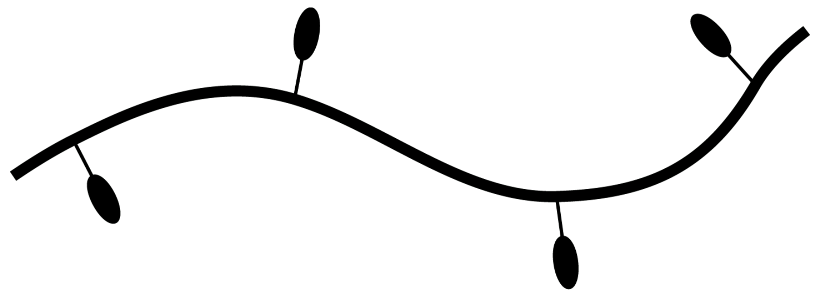
- Popularized in late 1960s in US Auto industry (GM)
  - R. Riesenfeld (1972)
  - W. Gordon
- Origin: the thin wood or metal strips used in building/ship construction

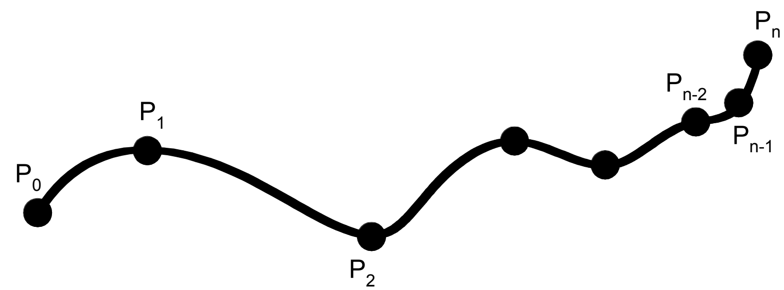
- Goal: define a curve as a set of piecewise simple polynomial functions connected together



Natural Splines

- Mathematical representation of physical splines
- C^2 continuous
- Interpolate all control points
- Have Global control (no local control)





B-splines: Basic Ideas

- Similar to Bézier curves
  - Smooth blending function times control points
- But:
  - Blending functions are non-zero over only a small part of the parameter range (giving us *local support*)
  - When nonzero, they are the "concatenation" of smooth polynomials. (They are piecewise!)

B-spline: Benefits

- User defines degree
  - Independent of the number of control points
- Produces a single piecewise curve of a particular degree
  - No need to stitch together separate curves at junction points
- Continuity comes for free

B-splines

- Defined similarly to Bézier curves
  - p_i are the control points
  - Computed with *basis functions* (*Basis-splines*)
    - B-spline basis functions are *blending functions*
  - Each point on the curve is defined by the *blending* of the control points (B_i is the *i-th* ***B-spline blending function***)

$$p(t) = \sum_{i=0}^{m} B_{i,d}(t)\, p_i$$

  - B_i is zero for most values of t!

B-splines: Cox-deBoor Recursion

- Cox-deBoor Algorithm: defines the blending functions for spline curves (not limited to deg 3)
  - curves are weighted avgs of lower degree curves
- Let $B_{i,d}(t)$ denote the *i*-th blending function for a B-spline of degree *d,* then:

$$B_{k,0}(t) = \begin{cases} 1, & \text{if } t_k \le t < t_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{k,d}(t) = \frac{t - t_k}{t_{k+d} - t_k} B_{k,d-1}(t) + \frac{t_{k+d+1} - t}{t_{k+d+1} - t_{k+1}} B_{k+1,d-1}(t)$$

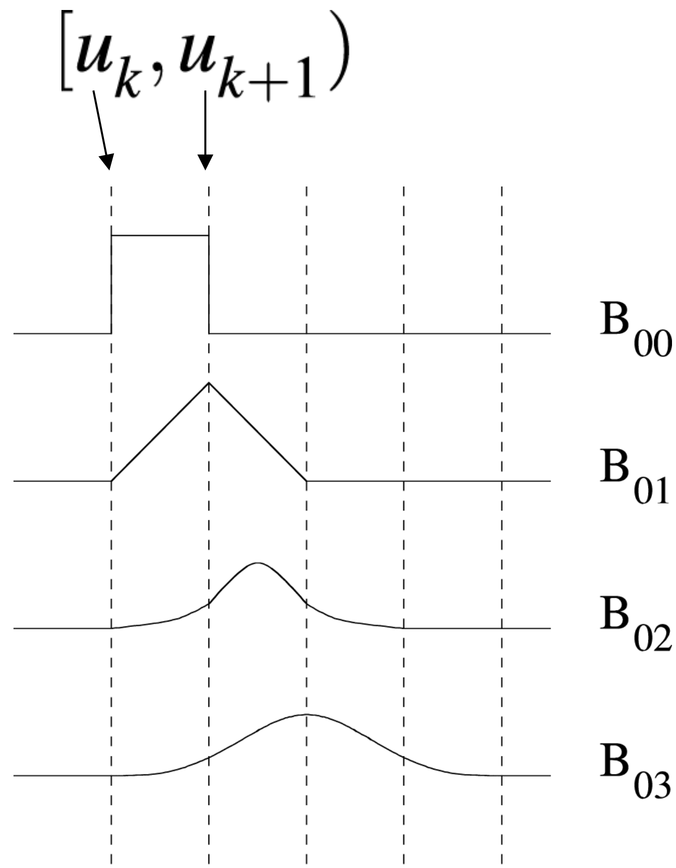
B-spline Blending Functions

$B_{k,0}(t)$ is a step function that is 1 in the interval

$B_{k,1}(t)$ spans two intervals and is a piecewise linear function that goes from 0 to 1 (and back)

$B_{k,2}(t)$ spans three intervals and is a piecewise quadratic that grows from 0 to 1/4, then up to 3/4 in the middle of the second interval, back to 1/4, and back to 0

$B_{k,3}(t)$ is a cubic that spans four intervals growing from 0 to 1/6 to 2/3, then back to 1/6 and to 0

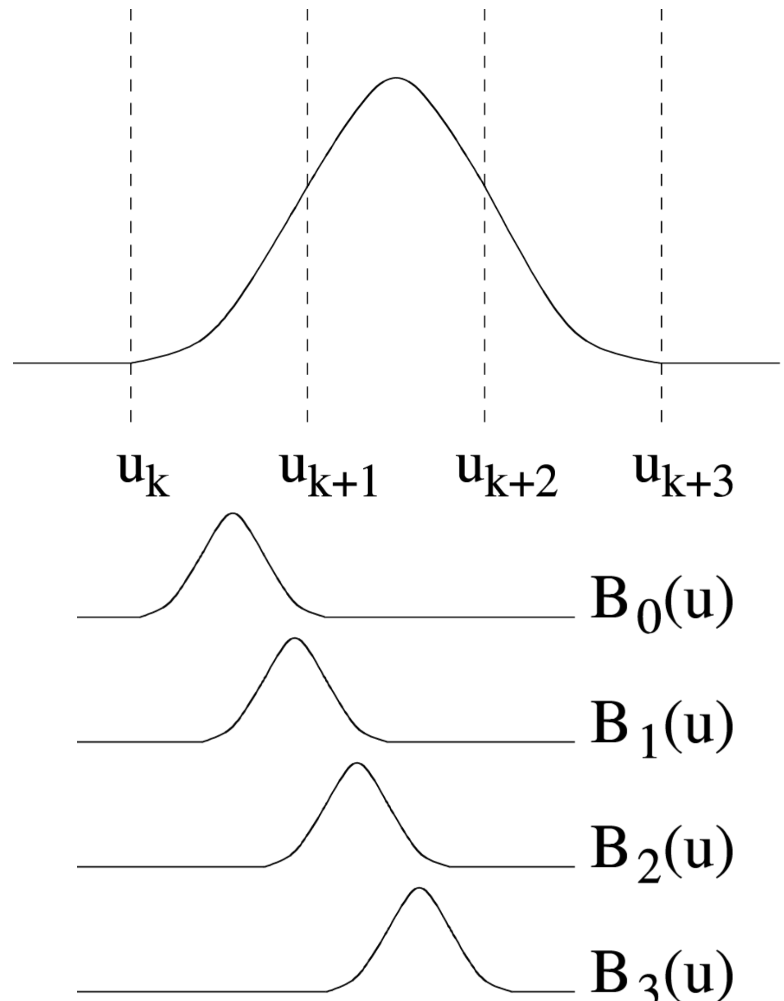


B-spline blending functions

Pics/Math courtesy of Dave Mount @ UMD-CP

B-spline Blending Functions: Example for 2nd Degree Splines

- Note: can't define a polynomial with these properties (both 0 and non-zero for ranges)
- Idea: subdivide the parameter space into *intervals* and build a *piecewise polynomial*
  - Each interval gets different polynomial function



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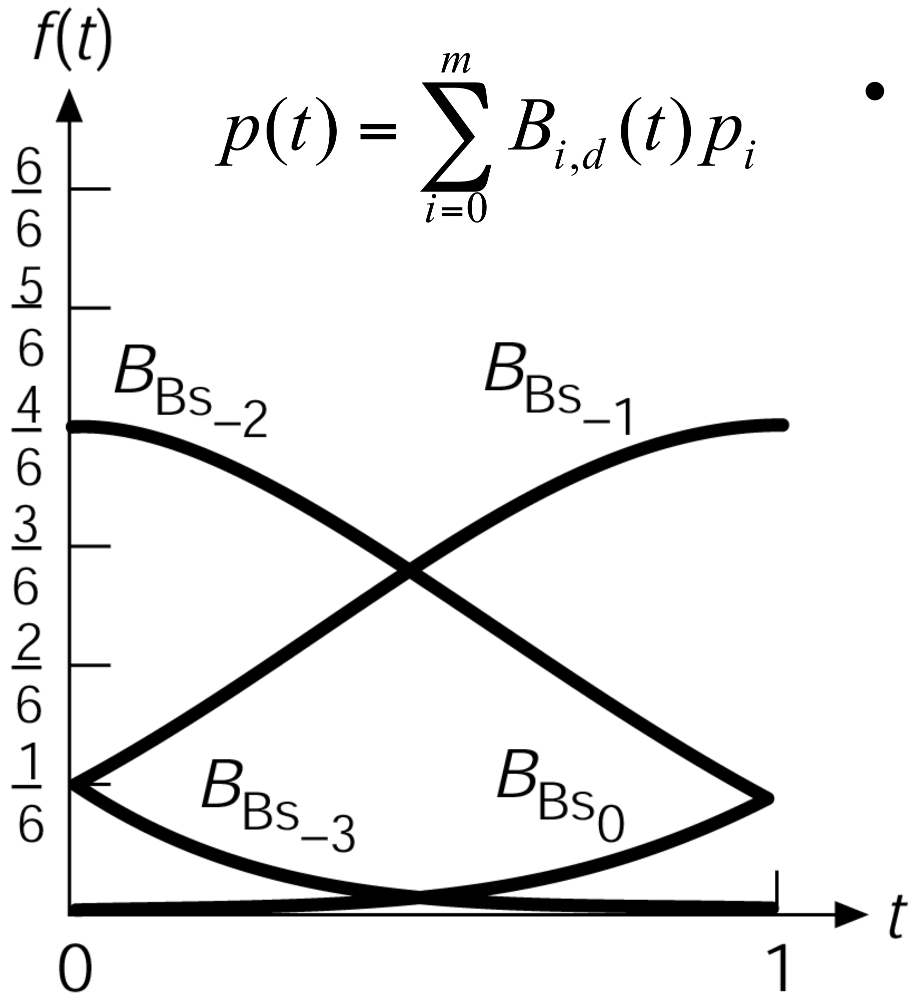
B-spline Blending Functions: Example for 3rd Degree Splines

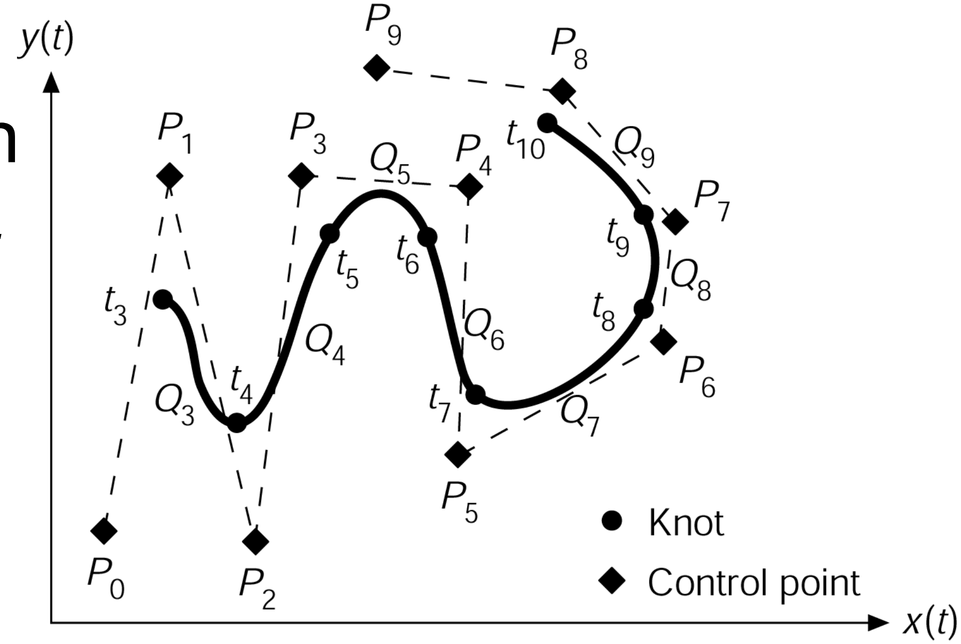
$$p(t) = \sum_{i=0}^{m} B_{i,d}(t)\, p_i$$



- Observe:
  - at t=0 and t=1 just four of the functions are non-zero
  - all are >=0 and sum to 1, hence the convex hull property holds for each curve segment of a B-spline

B-splines: Knot Selection

- Instead of working with the parameter space $0 \le t \le 1$, use $t_{\min} \le t_0 \le t_1 \le t_2 \ldots \le t_{m-1} \le t_{\max}$
- The ***knot points***
  - joint points between curve segments, Q_i
  - Each has a *knot value*
  - *m-1* knots for *m+1* points



1994 Foley/VanDam/Finer/Huges/Phillips ICG

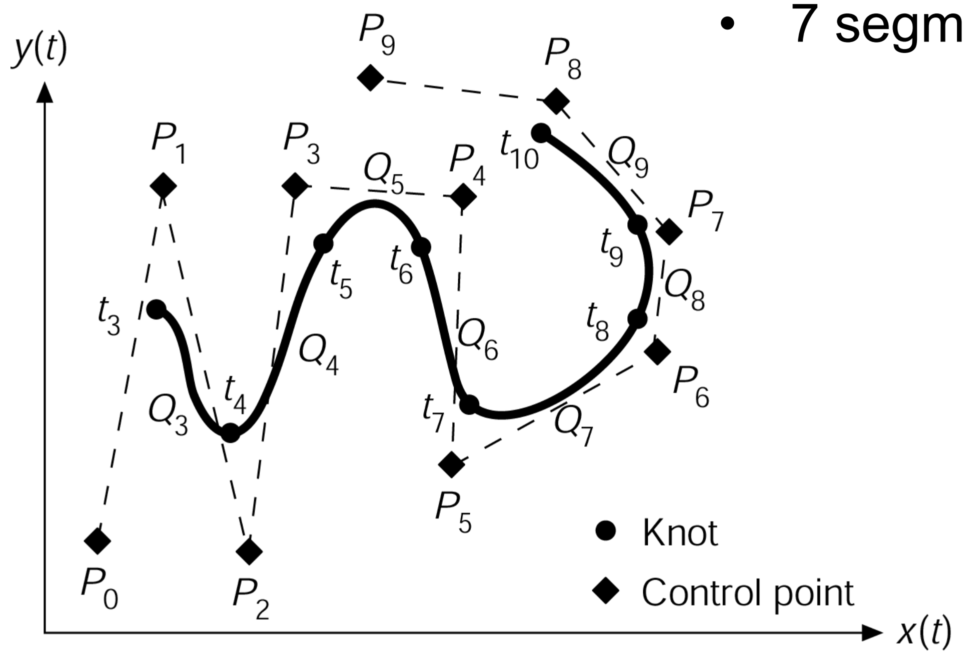
Uniform B-splines: Setting the Options

- Specified by
  - $m \geq 3$
  - *m+1* ***control points***, $P_0 \ldots P_m$
  - *m-2* ***cubic*** polynomial curve segments, $Q_3 \ldots Q_m$
  - *m-1* ***knot points***, $t_3 \ldots t_{m+1}$
  - ***segments*** Q_i of the B-spline curve are
    - defined over a knot interval $[t_i, t_{i+1}]$
    - defined by 4 of the control points, $P_{i-3} \ldots P_i$
  - segments Q_i of the B-spline curve are blended together into smooth transitions via (the new & improved) ***blending functions***

Example: Creating a B-spline

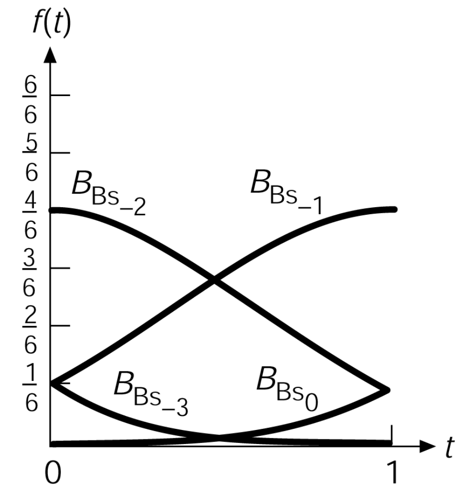
$$p(t) = \sum_{i=0}^{m} B_{i,d}(t)\, p_i$$

- m = 9
- 10 control points
- 8 knot points
- 7 segments



B-spline: Knot Sequences

- Even distribution of knots
  - *uniform* B-splines
  - Curve does not interpolate end points
    - first blending function not equal to 1 at t=0
- Uneven distribution of knots
  - *non-uniform* B-splines
  - Allows us to tie down the endpoints by repeating knot values (in Cox-deBoor, 0/0=0)
  - If a knot value is repeated, it increases the effect (weight) of the blending function at that point
  - If knot is repeated *d* times, blending function converges to 1 and the curve interpolates the control point



B-splines: Cox-deBoor Recursion

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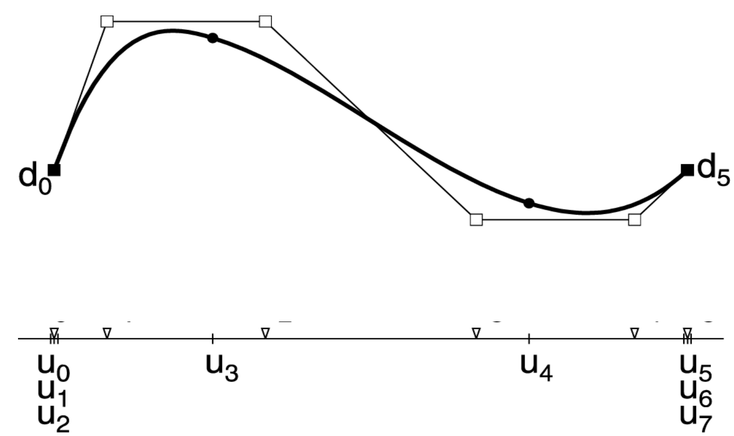
  - curves are weighted avgs of lower degree curves
- Let $B_{i,d}(t)$ denote the *i*-th blending function for a B-spline of degree *d,* then:

$$B_{k,0}(t) = \begin{cases} 1, & \text{if } t_k \le t < t_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{k,d}(t) = \frac{t - t_k}{t_{k+d} - t_k} B_{k,d-1}(t) + \frac{t_{k+d+1} - t}{t_{k+d+1} - t_{k+1}} B_{k+1,d-1}(t)$$

Creating a Non-Uniform B-spline: Knot Selection

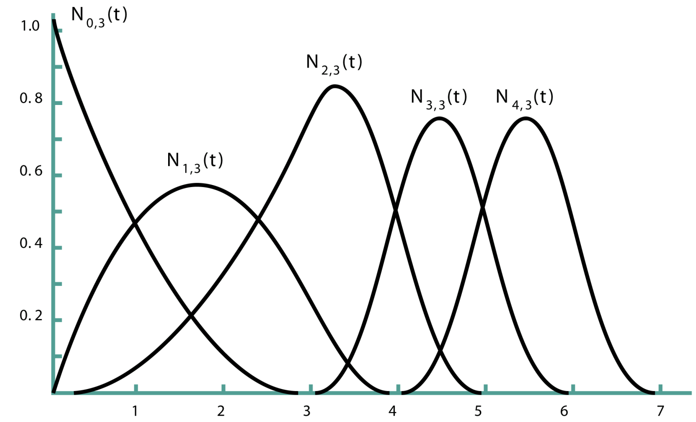
- Given curve of degree *d=3*, with *m+1* control points $\mathbf{p}_0, \ldots, \mathbf{p}_m$
  - first, create *m+d* knot values
  - use knot values *(0,0,0,1,2,…, m-2, m-1,m-1,m-1)* (adding two extra *0*'s and *m-1*'s)
  - Note
    - Causes Cox-deBoor to give added weight in blending to the first and last points when *t* is near t_{min} and t_{max}

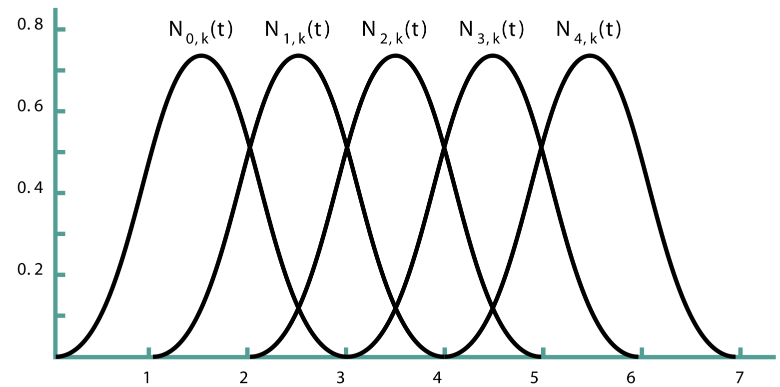


Pics/Math courtesy of G. Farin @ ASU

B-splines: Multiple Knots

- Knot Vector
  {0.0, 0.0, 0.0, 3.0, 4.0, 5.0, 6.0, 7.0}
- Several consecutive knots get the same value
- Changes the basis functions!

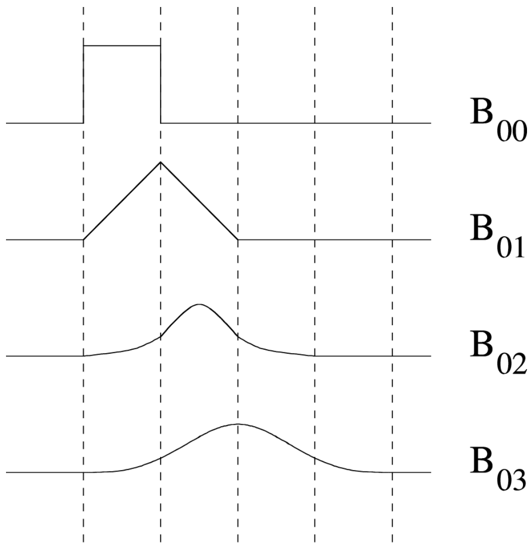




From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

$$p(t) = \sum_{i=0}^{m} B_{i,d}(t)\, p_i$$

B-spline Summary



$$B_{k,0}(t) = \begin{cases} 1, & \text{if } t_k \le t < t_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

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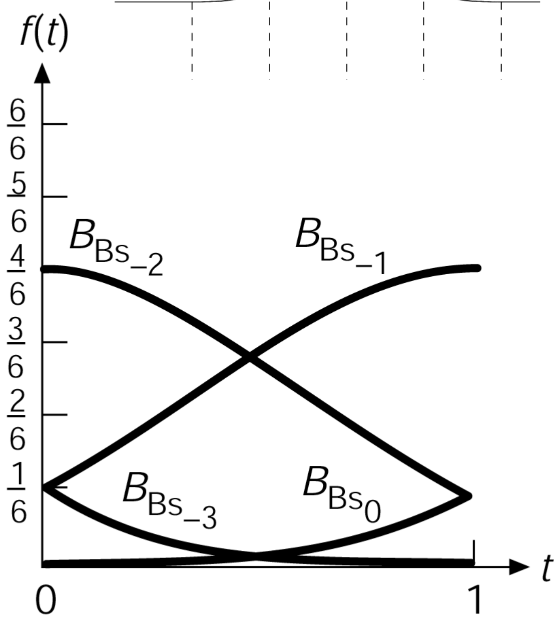
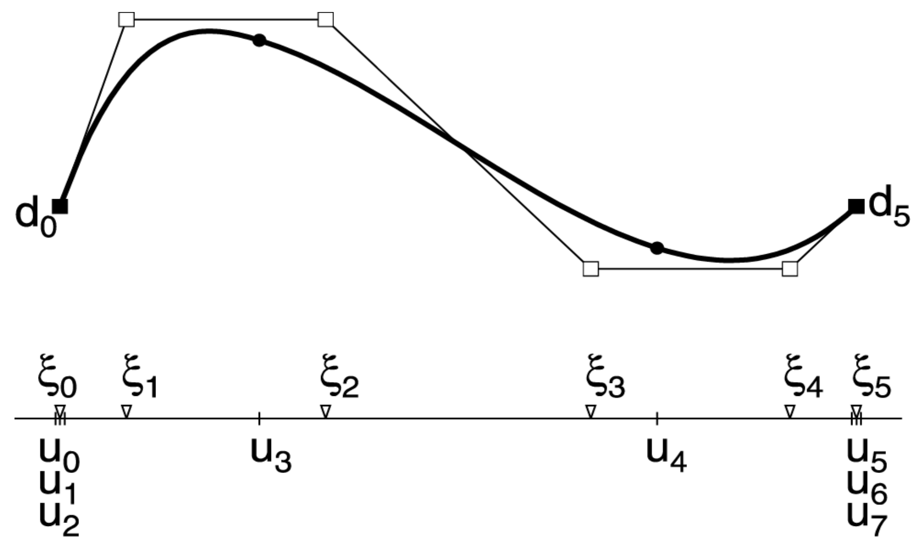
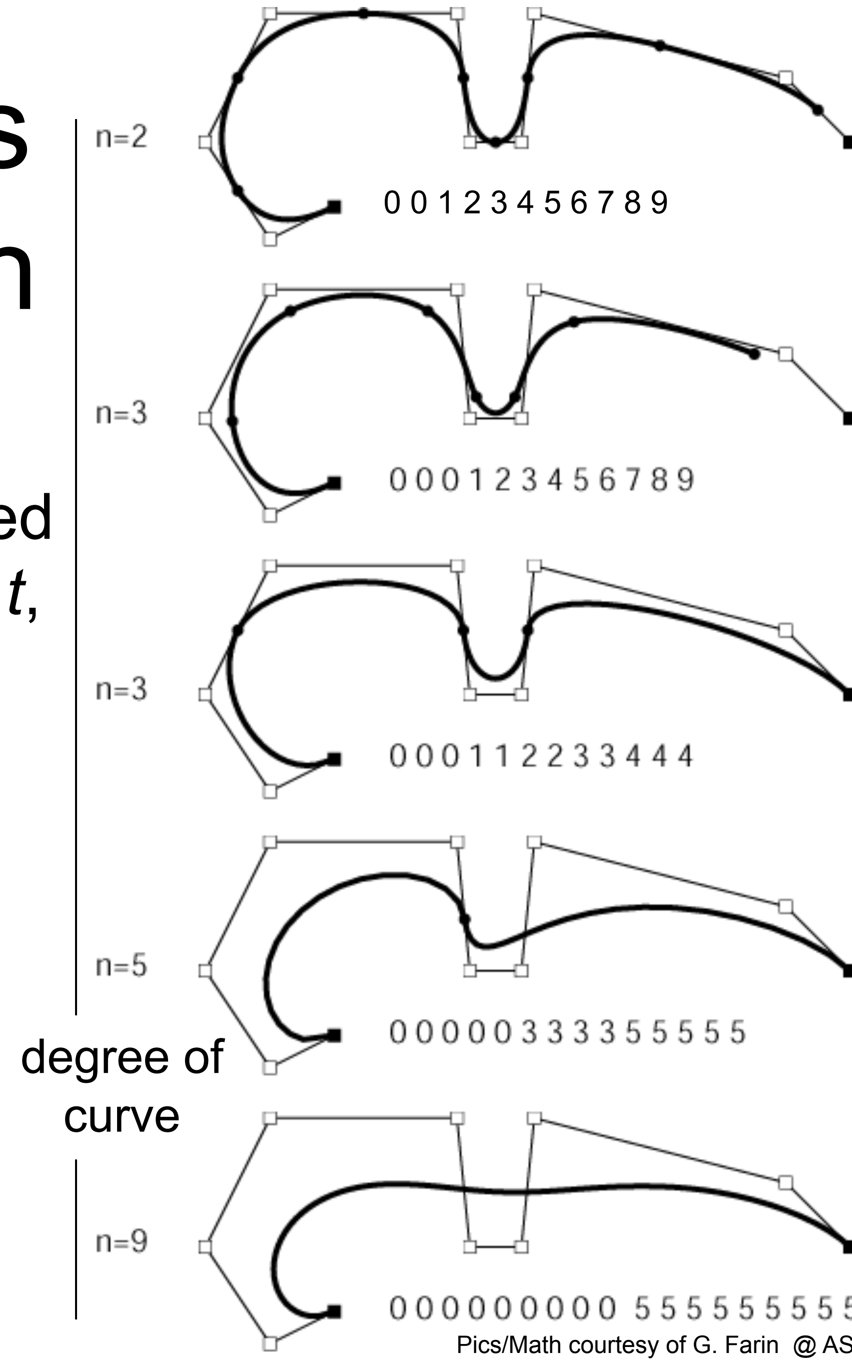




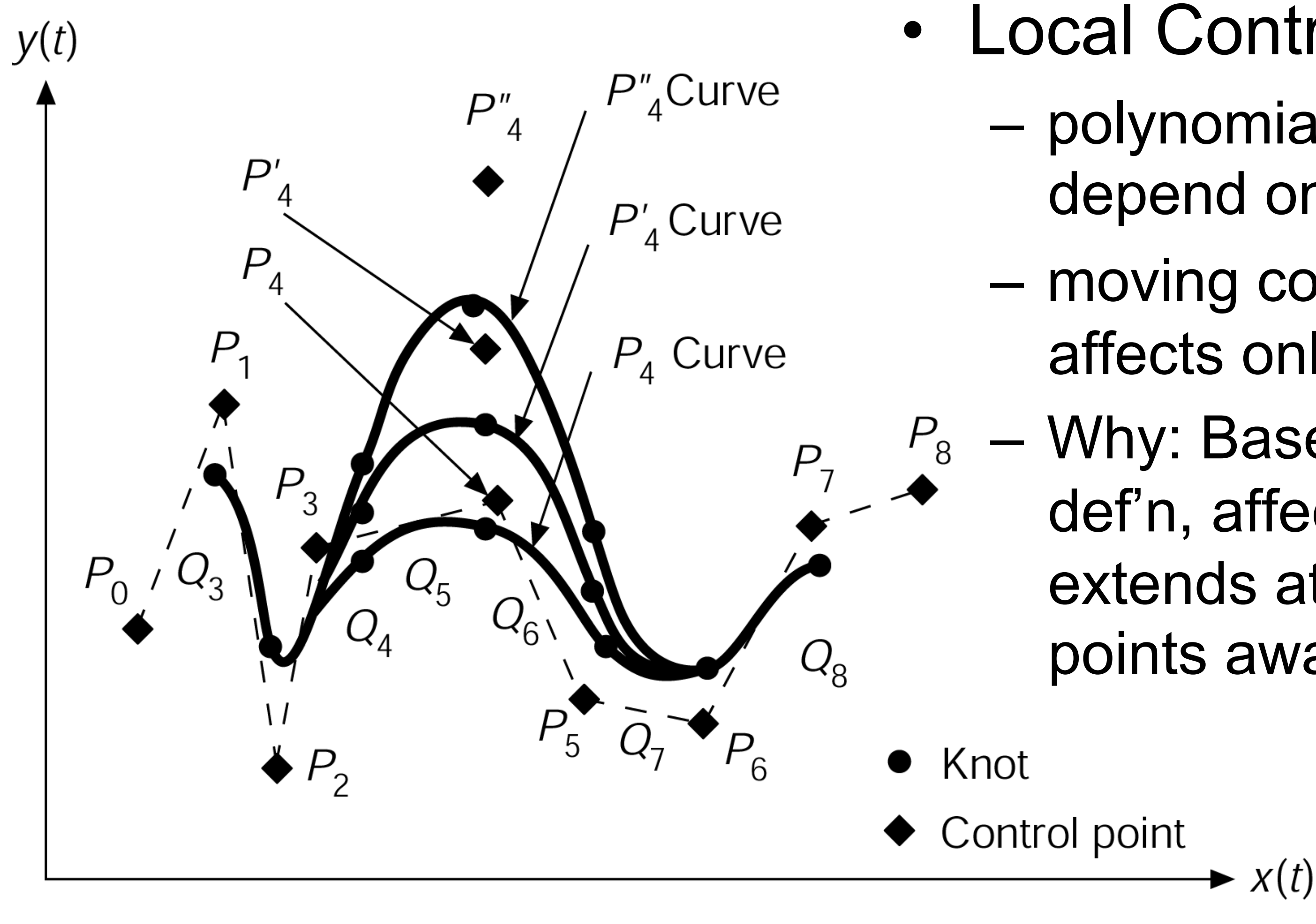
Watching Effects of Knot Selection

- 9 knot points (initially)
  - Note: knots are distributed parametrically based on *t*, hence why they "move"
- 10 control points
- Curves have as many segments as they have non-zero intervals in *u*



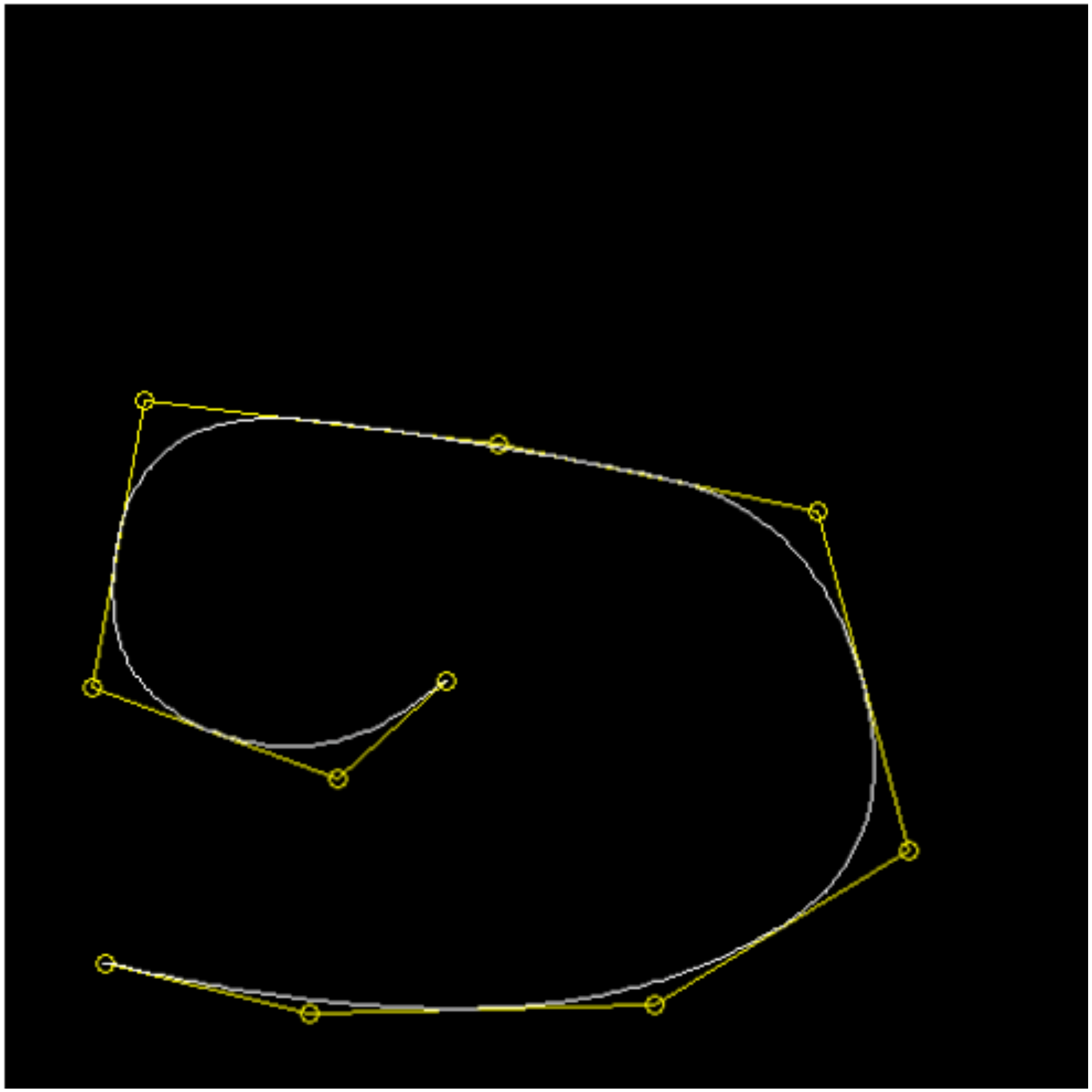
Pics/Math courtesy of G. Farin @ ASU

B-splines: Local Control Property



- Local Control
  - polynomial coefficients depend on a few points
  - moving control point (P_4) affects only local curve
  - Why: Based on curve def'n, affected region extends at most 2 knot points away

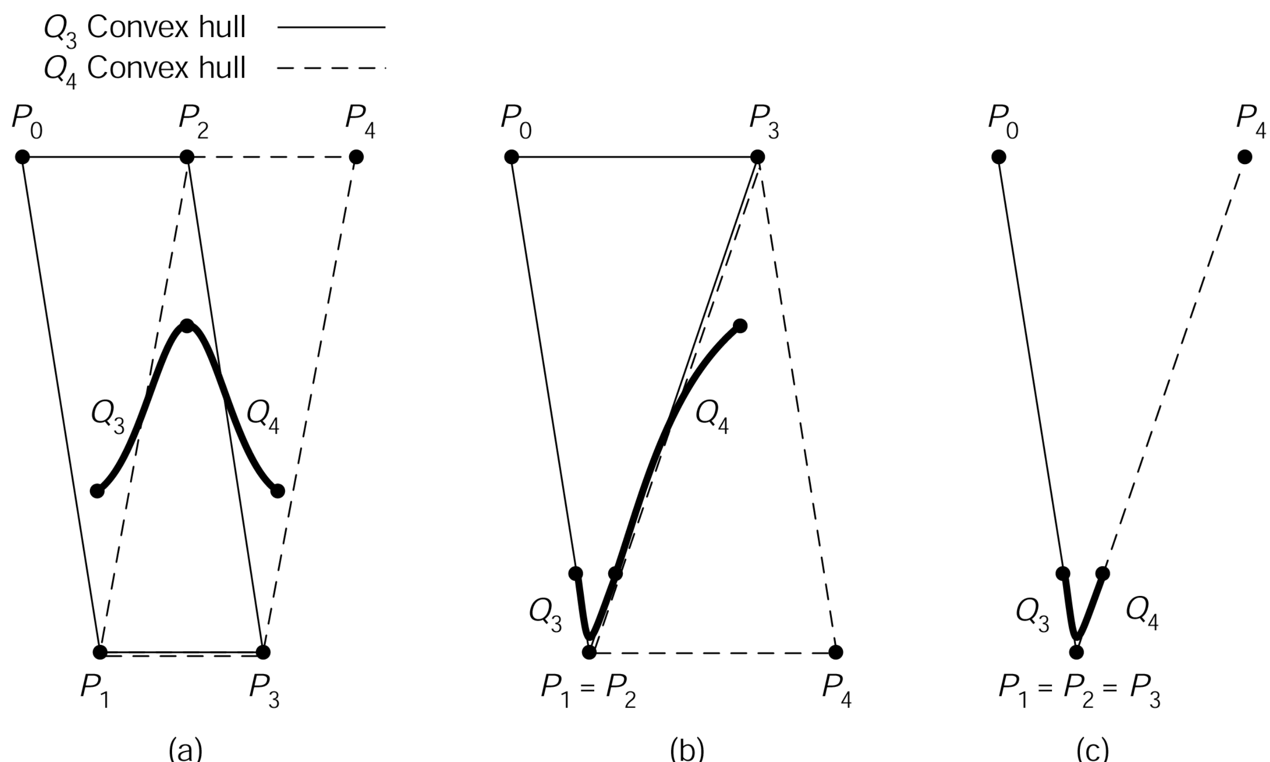
B-splines: Local Control Property



Recorded from: http://heim.ifi.uio.no/~trondbre/OsloAlgApp.html

B-splines: Convex Hull Property

- The effect of multiple control points on a uniform B-spline curve



B-splines: Continuity

- Derivatives are easy for cubics

$$p(u) = \sum_{k=0}^{3} u^k c_k$$

- Derivative:

$$p'(u) = c_1 + 2c_2 u + 3c_3 u^2$$

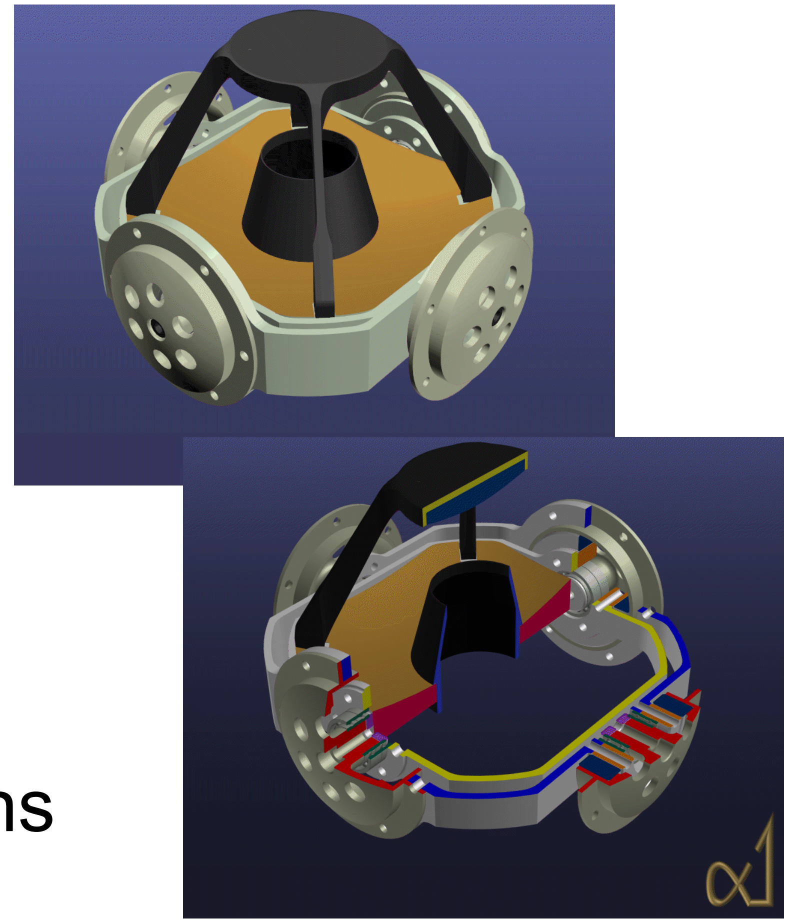
Easy to show C^0 , C^1 , C^2

B-splines: Setting the Options

- How to space the *knot points?*
  - **Uniform**
    - equal spacing of knots along the curve
  - **Non-Uniform**
- Which type of *parametric function*?
  - **Rational**
    - ***x(t), y(t), z(t)*** defined as ratio of cubic polynomials
  - **Non-Rational**

NURBS

- At the core of several modern CAD systems
  - I-DEAS, Pro/E, Alpha_1
- Describes analytic and freeform shapes
- Accurate and efficient evaluation algorithms
- Invariant under affine and perspective transformations



U of Utah, Alpha_1

Benefits of Rational Spline Curves

- Invariant under rotation, scale, translation, *perspective* transformations
  - transform just the control points, then regenerate the curve
  - (non-rationals only invariant under rotation, scale and translation)
- Can precisely define the conic sections and other analytic functions
  - conics require quadratic polynomials
  - conics only approximate with non-rationals

NURBS

Non-**u**niform **R**ational **B-spline**s: ***NURBS***

- Basic idea: four dimensional non-uniform B-splines, followed by normalization via homogeneous coordinates
  - If P_i is *[x, y, z, 1]*, results are invariant wrt perspective projection
- Also, recall in Cox-deBoor, knot spacing is arbitrary
  - knots are close together,
influence of some control points increases
  - Duplicate knots can cause points to interpolate
  - e.g. Knots = *{0, 0, 0, 0, 1, 1, 1, 1}* create a Bézier curve

Rational Functions

- Cubic curve segments

  $$x(t) = \frac{X(t)}{W(t)},\ y(t) = \frac{Y(t)}{W(t)},\ z(t) = \frac{Z(t)}{W(t)}$$

  where $X(t),\ Y(t),\ Z(t),\ W(t)$
  are all cubic polynomials with control points specified in homogenous coordinates, *[x,y,z,w]*
- Note: for 2D case, $Z(t) = 0$

Rational Functions: Example

- **Example:**
  - rational function: a *ratio* of polynomials
  - a rational parameterization in u of a unit circle in xy-plane:

$$\begin{aligned} x(u) &= \frac{1-u^2}{1+u^2} \\ y(u) &= \frac{2u}{1+u^2} \\ z(u) &= 0 \end{aligned}$$

  - a unit circle in 3D homogeneous coordinates:

$$\begin{aligned} x(u) &= 1-u^2 \\ y(u) &= 2u \\ z(u) &= 0 \\ w(u) &= 1+u^2 \end{aligned}$$

NURBS: **Notation Alert**

- Depending on the source/reference
  - Blending functions are either $B_{i,d}(u)$ or $N_{i,d}(u)$
  - Parameter variable is either *u* or *t*
  - Curve is either *C* or *P* or *Q*
  - Control Points are either P_i or B_i
  - Variables for order, degree, number of control points etc are frustratingly inconsistent
    - *k, i, j, m, n, p, L, d, ….*

NURBS: **Notation Alert**

1. If defined using *homogenous coordinates*, the 4th (3rd for 2D) dimension of each P_i is the weight

2. If defined as *weighted euclidian*, a separate constant w_i, is defined for each control point

NURBS

- A *d*-th degree NURBS curve *C* is def'd as:

$$C(u) = \frac{\sum_{i=0}^{n-1} w_i B_{i,d}(u) P_i}{\sum_{i=0}^{n-1} w_i B_{i,d}(u)}$$

Where

- control points, P_i
- *d*-th degree B-spline blending functions, $B_{i,d}(u)$
- the *weight*, w_i, for control point P_i
(when all $w_i = 1$, we have a B-spline curve)

Observe: Weights Induce New Rational Basis Functions, *R*

- Setting:

$$R_i(u) = \frac{w_i B_{i,d}(u)}{\sum_{i=0}^{n-1} w_i B_{i,d}(u)}$$

Allows us to write: $C(u) = \sum_{i=0}^{n-1} R_{i,d}(u) P_i$

Where $R_{i,d}(u)$ are *rational basis functions*

– piecewise rational basis functions on $u \in [0,1]$

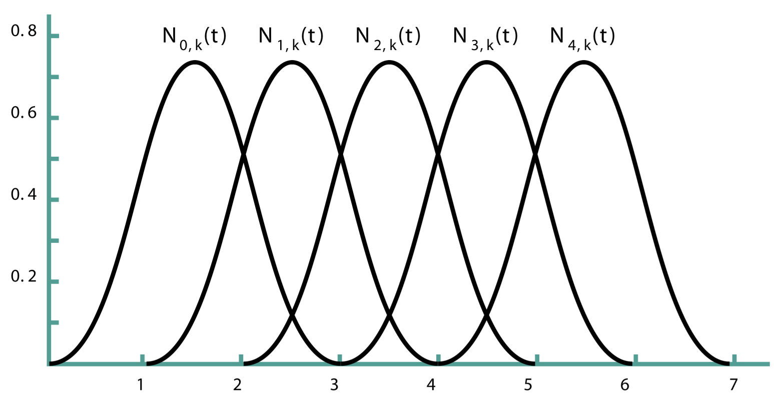
– weights are incorporated into the basis fctns

Geometric Interpretation of NURBS

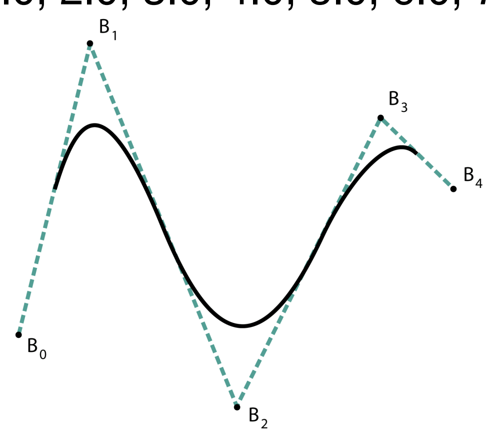
- With Homogeneous coordinates, a rational *n*-D curve is represented by polynomial curve in *(n+1)*-D
- Homogeneous 3D control points are written as: $P_i^w = w_i x_i, w_i y_i, w_i z_i, w_i$ in 4D where $w \neq 0$
- To get P_i, divide by w_i
  - a perspective transform with center at the origin
- Note: weights can allow final curve shape to go outside the convex hull (i.e. negative *w*)

NURBS: Examples

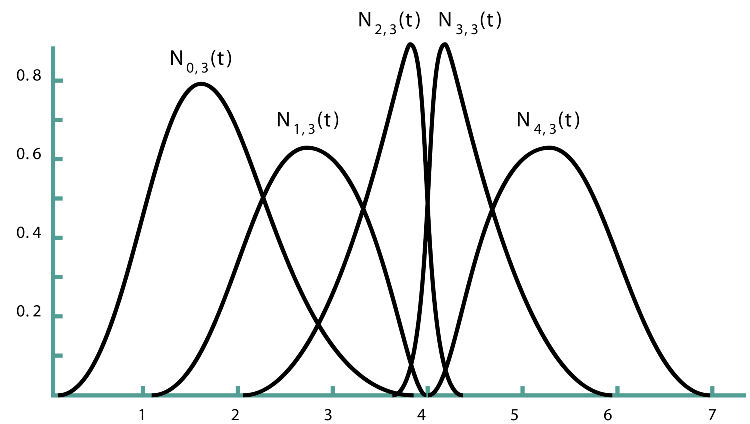
- Unif. Knot Vector



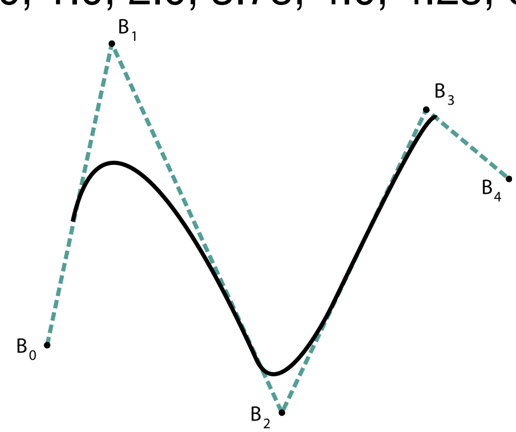
{0.0, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0}



- Non-Unif. Knot Vector



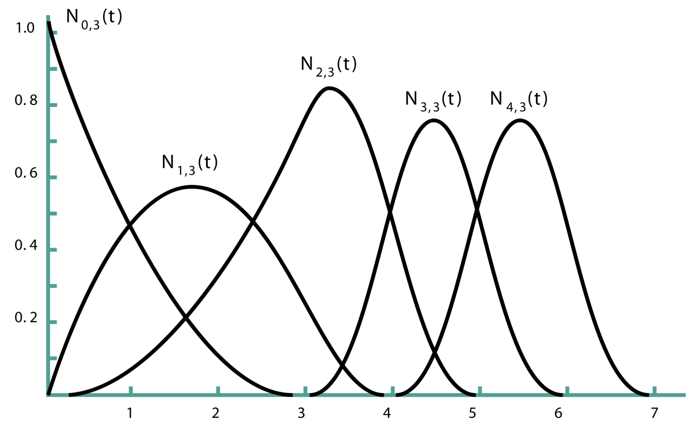
{0.0, 1.0, 2.0, 3.75, 4.0, 4.25, 6.0, 7.0}

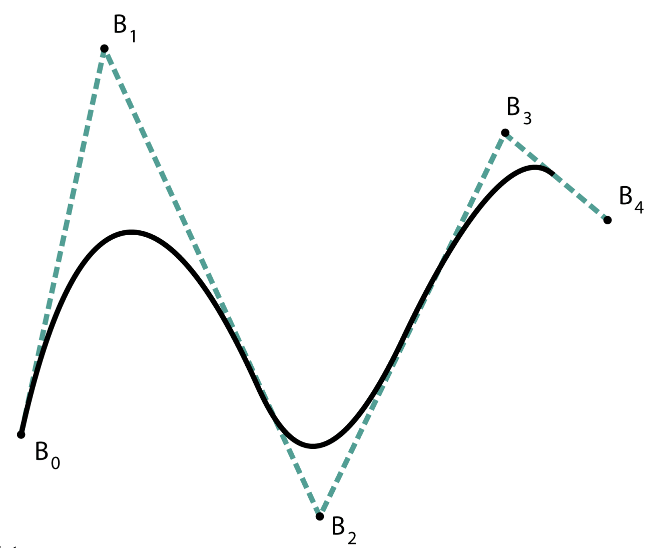


From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

NURBS: Examples

- Knot Vector
  {0.0, 0.0, 0.0, 3.0, 4.0, 5.0, 6.0, 7.0}
- Several consecutive knots get the same value
- Bunches up the curve and forces it to interpolate





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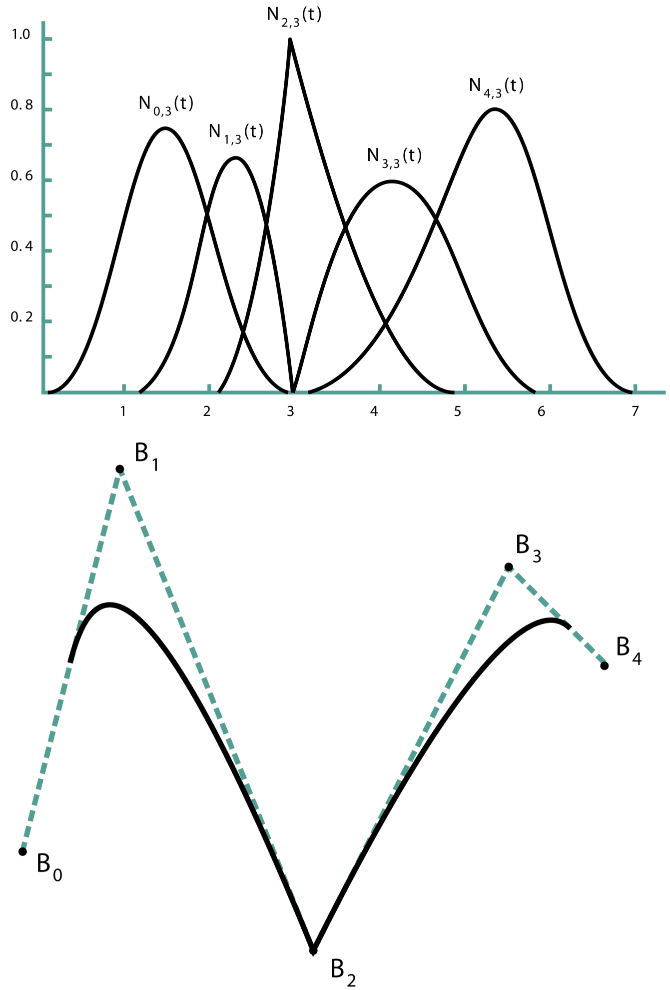
NURBS: Examples

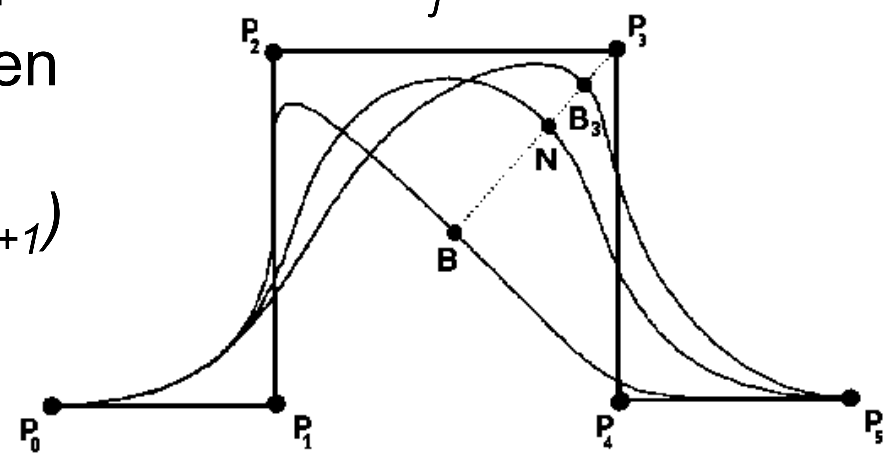


- Knot Vector
  {0.0, 1.0, 2.0, **3.0, 3.0**, 5.0, 6.0, 7.0}
- Several consecutive knots get the same value
- Bunches up the curve and forces it to interpolate
- Can be done midcurve

From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

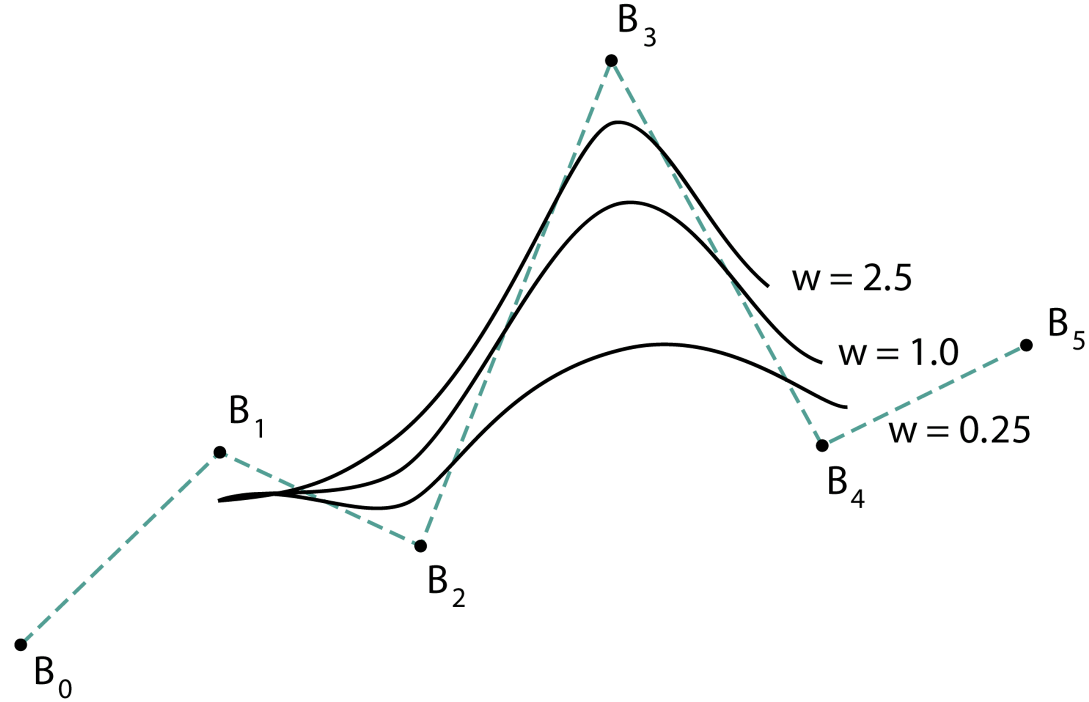
The Effects of the Weights

- w_i of P_i effects only the range $[u_i, u_{i+k+1})$
- If $w_i=0$ then P_i does not contribute to C
- If w_i increases, point B and curve C are *pulled toward* P_i and pushed away from P_j
- If w_i decreases, point B and curve C are *pushed away* from P_i and pulled toward P_j
- If w_i approaches infinity then B approaches 1 and B_i -> P_i , if u in $[u_i, u_{i+k+1})$



The Effects of the Weights

- Increased weight pulls the curve toward B_3



From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Programming assignment 3

- Input PostScript-like file containing polygons
- Output B/W XPM
- Implement viewports
- Use Sutherland-Hodgman intersection for polygon clipping
- Implement scanline polygon filling. *(You cannot use flood filling)*